Functions having quadratic differences in a given class

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Abstract. Starting from a problem of Z. Daróczy we define the quadratic difference property and show that the class of all real-valued continuous functions on $\mathbf{R}$ and some of its subclasses have this property while the class of all bounded functions does not have.

1. Introduction

For a function $f: \mathbf{R} \to \mathbf{R}$ (the reals) and for a fixed $y \in \mathbf{R}$ define the function $\Delta_y f$ on $\mathbf{R}$ by $\Delta_y f(x) = f(x+y) - f(x)$, $x \in \mathbf{R}$. The functions $A, N: \mathbf{R} \to \mathbf{R}$ are said to be additive and quadratic if

$$A(x+y) = A(x) + A(y) \qquad x, y \in \mathbf{R}$$

and

$$N(x+y) + N(x-y) = 2N(x) + 2N(y) \qquad x, y \in \mathbf{R},$$

respectively. It is well-known (see [1], [5], [2]) that, if an additive function is bounded from one side on an interval of positive length then $A(x) = cx$, $x \in \mathbf{R}$ for some $c \in \mathbf{R}$ and there are discontinuous additive functions. Similarly, if a quadratic function is bounded on an interval of positive length then $N(x) = dx^2$, $x \in \mathbf{R}$ for some $d \in \mathbf{R}$ and there are discontinuous quadratic functions.

In [4] Z. Daróczy asked that for which properties T the following statement is true:

($*$) Let $f: \mathbf{R} \to \mathbf{R}$ be a function such that for all fixed $y \in \mathbf{R}$ the function $\Delta_y \Delta_{-y} f$ has the property T. Then

$$f = f^* + N + A \qquad \text{on} \quad \mathbf{R} \tag{1}$$

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where $f^\star$ has the property T, N is a quadratic function and A is an additive function.

In this note we prove that, if T is the k-times continuously differentiability ($k \geq 0$ integer) or k-times differentiability ($k > 0$ integer or $k = +\infty$) or being polynomial then the statement $(*)$ is true while if T is the boundedness then $(*)$ is not true.

2. Preliminary results

The following lemma will play an important role in our investigations.

Lemma 1. *For all functions $f\colon \mathbf{R} \to \mathbf{R}$ and for all $u, v, x \in \mathbf{R}$ we have*

$$
\begin{aligned}
\Delta_u \Delta_v f(x) &= \Delta_{\frac{u-v}{2}} \Delta_{-\frac{u-v}{2}} f\left(x + \frac{u+v}{2}\right) \\
&\quad - \Delta_{\frac{u+v}{2}} \Delta_{-\frac{u+v}{2}} f\left(x + \frac{u+v}{2}\right).
\end{aligned}
\tag{2}
$$

The proof is a simple computation therefore it is omitted.

An other basic tool we will use is the following result of DE BRUIJN ([3] Theorem 1.1.)

Theorem 1. *Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is a function such that the function $\Delta_y f$ is continuous for all fixed $y \in \mathbf{R}$. Then $f = f^\star + A$ on $\mathbf{R}$ with some continuous $f^\star\colon \mathbf{R} \to \mathbf{R}$ and additive $A\colon \mathbf{R} \to \mathbf{R}$.*

Finally we will need the following two lemmata.

Lemma 2. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function such that $\Delta_u \Delta_v f$ is continuous for all fixed $u, v \in \mathbf{R}$. Define*

$$
H(x,u,v) = \Delta_u \Delta_v f(x) - f(u+v) + f(u) + f(v) \qquad x, u, v \in \mathbf{R}. \tag{3}
$$

Then the function $(x,u) \to H(x,u,v)$, $(x,u) \in \mathbf{R}^2$ is continuous for all fixed $v \in \mathbf{R}$.

Proof. Let $v \in \mathbf{R}$ be fixed. Since $\Delta_u(\Delta_v f)$ is continuous for all fixed $u \in \mathbf{R}$, Theorem 1 implies that $\Delta_v f = f_v^\star + A_v$ on $\mathbf{R}$ where $f_v^\star\colon \mathbf{R} \to \mathbf{R}$ is continuous and A_v is additive. Thus, by (3),

$$
\begin{aligned}
H(x,u,v) &= \Delta_v f(x+u) - \Delta_v f(x) - \Delta_v f(u) + f(v) \\
&= f_v^\star(x+u) - f_v^\star(x) - f_v^\star(u) + f(v)
\end{aligned}
$$

whence the continuity of $(x,u) \to H(x,u,v)$, $(x,u) \in \mathbf{R}^2$ follows.

Lemma 3. *Suppuse that L is one of the classes of the real-valued functions defined on $\mathbf{R}$ which are k-times continuously differentiable for some $k \geq 0$ integer or k-times differentiable for some $1 \leq k \leq +\infty$ or polynomials. If $f: \mathbf{R} \to \mathbf{R}$ is continuous and $\Delta_y f \in L$ for all fixed $y \in \mathbf{R}$ then $f \in L$.*

Proof. If L is the class of the continuous functions ($k = 0$) or of the polynomials, furthermore f is continuous and $\Delta_y f \in L$ for all fixed $y \in \mathbf{R}$ then, by Theorem 1 and by [3] page 203, respectively, $f = f^\star + A$ for some $f^\star \in L$ and additive function A. Therefore, by continuity of f, $A(x) = cx$, $x \in \mathbf{R}$ with some $c \in \mathbf{R}$ whence $f \in L$ follows.

The remaining statement of Lemma 3 is just Lemma 3.1. in [3].

3. The main results

For the formulation of our main results let us begin with the following

Definition. A subset E of the set of all functions $f: \mathbf{R} \to \mathbf{R}$ is said to have the quadratic difference property if for all $f: \mathbf{R} \to \mathbf{R}$, with $\Delta_y \Delta_{-y} f \in E$ for all $y \in \mathbf{R}$, the decomposition (1) holds true on $\mathbf{R}$ where $f^\star \in E$, N is a quadratic function and A is an additive function.

First we prove the following

Theorem 2. *The class of all continuous functions $f: \mathbf{R} \to \mathbf{R}$ has the quadratic difference property.*

Proof. By (2) in Lemma 1 we have that $\Delta_u \Delta_v f$ is continuous for all fixed $u, v \in \mathbf{R}$. In particular, $\Delta_u(\Delta_1 f)$ is continuous for all fixed $u \in \mathbf{R}$. Applying Theorem 1 to $\Delta_1 f$ we have

$$\Delta_1 f = f_0 + a \qquad \text{on } \mathbf{R} \tag{4}$$

with some continuous $f_0: \mathbf{R} \to \mathbf{R}$ and additive $a: \mathbf{R} \to \mathbf{R}$. Define the function B on $\mathbf{R}^2$ by

$$B(u,v) = \int_0^1 \Delta_u \Delta_v f - \int_0^{u+v} f_0 + \int_0^u f_0 + \int_0^v f_0, \qquad (u,v) \in \mathbf{R}^2. \tag{5}$$

Obviously, B is symmetric. Now we show that B is additive in its first variable. For all u, t and v, we have

$$B(u+t,v) - B(u,v) = \int_0^1 \Delta_{u+t} \Delta_v f - \int_0^{u+t+v} f_0 + \int_0^{u+t} f_0 + \int_0^v f_0$$

$$- \int_0^1 \Delta_u\Delta_v f + \int_0^{u+v} f_0 - \int_0^u f_0 - \int_0^v f_0$$
$$= \int_u^{u+1} \Delta_t\Delta_v f - \int_0^{u+t+v} f_0 + \int_0^{u+t} f_0 + \int_0^{u+v} f_0 - \int_0^u f_0.$$

Since $\Delta_t\Delta_v f$ and f_0 are continuous functions, the right hand side is continuously differentiable with respect to u then so is the left hand side. Differentiating both sides with respect to u and taking into consideration (4) we obtain that

$$\begin{aligned}\frac{\partial}{\partial u}[B(u+t,v) - B(u,v)] &= \Delta_t\Delta_v\Delta_1 f(u) - f_0(u+t+v) + f_0(u+t) \\ &\quad + f_0(u+v) - f_0(u) \\ &= \Delta_t\Delta_v(f_0 + a)(u) - \Delta_t\Delta_v f_0(u) \\ &= \Delta_t\Delta_v a(u) = 0 \qquad (a \text{ being additive}).\end{aligned}$$

Therefore

$$B(u+t,v) - B(u,v) = B(t,v) - B(0,v) = B(t,v),$$

that is, B is additive in its first (and by the symmetry also in its second) variable. Thus, it is well-known (see [2]) and easy to see that, the function $N: \mathbf{R} \to \mathbf{R}$ defined by $N(u) = \frac{1}{2}B(u,u)$, $u \in \mathbf{R}$ is quadratic and

$$B(u,v) = N(u+v) - N(u) - N(v) \qquad u,v \in \mathbf{R}. \tag{6}$$

Define the function $H: \mathbf{R}^3 \to \mathbf{R}$ by (3) and apply Lemma 2 to get the continuity of the function $(x,u) \to H(x,u,v)$, $(x,u) \in \mathbf{R}^2$ for all fixed $v \in \mathbf{R}$. This implies that the function $s: \mathbf{R}^2 \to \mathbf{R}$ defined by

$$s(u,v) = \int_0^1 H(x,u,v)dx \qquad (u,v) \in \mathbf{R}^2$$

is continuous in its first variable (for all fixed $v \in \mathbf{R}$). Therefore, by (3), (5) and (6) we have

$$s(u,v) = \int_0^1 \Delta_u\Delta_v f - f(u+v) + f(u) + f(v)$$

$$
\begin{aligned}
&= B(u,v) + \int_0^{u+v} f_0 - \int_0^{u} f_0 - \int_0^{v} f_0 - f(u+v) + f(u) + f(v) \\
&= N(u+v) - f(u+v) - (N(u) - f(u)) - (N(v) - f(v)) \\
&\quad + \int_0^{u+v} f_0 - \int_0^{u} f_0 - \int_0^{v} f_0 \\
&= -\Delta_v(f-N)(u) - (N(v) - f(v)) + \int_0^{u+v} f_0 - \int_0^{u} f_0 - \int_0^{v} f_0 .
\end{aligned}
$$

This implies that $\Delta_v(f-N)$ is continuous for all fixed $v \in \mathbf{R}$ and Theorem 1 can be applied again to get the decomposition $f - N = f^\star + A$ on $\mathbf{R}$ with some continuous $f^\star : \mathbf{R} \to \mathbf{R}$ and additive function A, that is, (1) holds and the proof is complete.

Theorem 3. *Let L be as in Lemma 3. Then L has the quadratic difference property.*

Proof. If L is the class of all continuous functions then the statement is proved by Theorem 2. In the remaining cases, since all functions in L are continuous, Theorem 2 implies the decomposition (1) with continuous $f^\star$, quadratic N and additive A. We now prove that $f^\star \in L$. For all $y \in \mathbf{R}$ we get from (1) that

$$\Delta_y \Delta_{-y} f = \Delta_y \Delta_{-y} f^\star + 2N(y). \tag{7}$$

Therefore $\Delta_y \Delta_{-y} f^\star \in L$ for all fixed $y \in \mathbf{R}$. Applying (2) in Lemma 1 we obtain that $\Delta_u(\Delta_v f^\star) \in L$ for all fixed $u, v \in \mathbf{R}$. Obviously $\Delta_v f^\star$ is continuous thus, by Lemma 3, $\Delta_v f^\star \in L$. Since $f^\star$ is continuous, Lemma 3 can be applied again to get $f^\star \in L$.

Remark. The set of all bounded functions $f: \mathbf{R} \to \mathbf{R}$ does not have the quadratic difference property. Indeed, let

$$f(x) = x \ln(x^2+1) + 2 \operatorname{arc\,tg} x - 2x, \qquad x \in \mathbf{R}.$$

Applying the Lagrangian mean value theorem with fixed $u, v, x \in \mathbf{R}$ we have

$$\Delta_u \Delta_v f(x) = u \Delta_v f'(\xi) = uv f''(\eta) \tag{8}$$

for some $\xi, \eta \in \mathbf{R}$. Since $|f''(\eta)| = \frac{2|\eta|}{\eta^2+1} \leq 1$, (8) implies that $|\Delta_y \Delta_{-y} f(x)| \leq y^2$ for all $x, y \in \mathbf{R}$, that is, $\Delta_y \Delta_{-y} f$ is bounded for all fixed $y \in \mathbf{R}$. Suppose that f has the decomposition (1) for some bounded $f^\star : \mathbf{R} \to \mathbf{R}$, quadratic N and additive A. Then $N + A$ must be bounded on any bounded interval. Thus $N(x) + A(x) = \alpha x^2 + \beta x$, $x \in \mathbf{R}$ for some $\alpha, \beta \in \mathbf{R}$. This and (1) imply that

$$f^\star(x) = x \ln(x^2+1) + 2 \operatorname{arc\,tg} x - \alpha x^2 - (2+\beta)x, \qquad x \in \mathbf{R}. \tag{9}$$

Since $f^\star$ is bounded, $0 = \lim\limits_{x \to +\infty} \frac{f^\star(x)}{x^2} = -\alpha$ and thus

$$0 = \lim_{x \to +\infty} \frac{f^\star(x) - 2 \operatorname{arc\,tg} x}{x} = \lim_{x \to +\infty} \left(\ln(x^2+1) - (2+\beta) \right),$$

which is a contradiction. This shows that the set of all bounded functions does not have the quadratic difference property.

References

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